

Diagnosing Contextuality in a Topological Quantum Field Theory: a Toy Model

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Abstract

This paper synthesizes the previously developed modal contexts with quantum contextuality. We reformulate the Kochen-Specker theorem in terms of cobordisms. Finally, we provide a concrete example of a $(1 + 1)$ -dimensional topological quantum field theory as a toy model. It is shown that the Kochen-Specker theorem is equivalent to this special type of TQFT.

1 Introduction

As a well-known consequence of the Kochen–Specker theorem [4, 7], quantum theory is inherently contextual: in certain measurement scenarios, it is impossible to assign a global section to the presheaf of outcomes consistent with all local contexts. It has been demonstrated [3] that non-contextual scenarios correspond to acyclic measurement graphs—i.e., trees—while contextuality arises precisely when the measurement graph contains non-trivial cycles. That is, contextuality manifests as non-vanishing first homology $H_1(\Gamma)$ of the graph $\Gamma = \mathcal{S}_{\text{Graph}}$, thereby witnessing an obstruction to the existence of a global assignment.

We shall now recall the definition of a “scenario:”

Definition 1.1 ([5, 6]). *A (measurement, contextuality, compatibility) scenario $\mathcal{S}$ is a pair $(\mathcal{X}, \mathcal{C})$ with $\mathcal{X}$ a collection of measurements and $\mathcal{C}$ a compatibility cover of $\mathcal{X}$.*

After some mild re-interpretation, we can recast this in the language of [8] to say:

Definition 1.2. *Let $\mathcal{C}$ be a maximal context, so that for every measurement $\mathcal{M}$ on an orbifold $\mathcal{O}$, there exists a Heyting algebra $\mathfrak{H}$ such that for every connected subspace $\mathcal{O}_i \subseteq \mathcal{O}$, there exists no extension $\mathfrak{H}^+ \succ \mathfrak{H}$ such that $\mathcal{M}_{\mathcal{O}_i} = \mathfrak{h}$ for any $\mathfrak{h} \in \mathfrak{H}^+ \setminus \mathfrak{H}$. Then, a scenario is precisely the pair $(\mathcal{M}, \mathcal{C})$, where $\mathcal{M}$ is the product*

$$\mathcal{M} = \prod_i \mathcal{M}_{\mathcal{O}_i}$$

In other words, a scenario is just a collection of maps from a “maximal”¹ context $\mathcal{C}$ to some orbifold $\mathcal{O}$. In other words, it is a *fiber bundle in disguise*. Namely, it is the bundle

$$\pi : \mathcal{C} \rightarrow \bigcup_i \mathcal{O}_i$$

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¹Maximal, in this case, with respect to the coverage of the Heyting algebra $\mathfrak{H}$ by a collection $\mathcal{V}$ of Grothendieck universes which share the same top element $\top$

and the Kochen-Specker theorem tells us that for a point $p_i \in \mathcal{O}_i$, the *fiber over the point*, $\pi^{-1}(p_i)$ fails to be dense if and only if p_i is outside the measurement scenario $\mathcal{S}$. This seems to suggest that the map:

$$\mathcal{E} : \mathfrak{H} \rightarrow \mathfrak{H}^+$$

introduces an obstruction² to the sheafification of $\pi^{\mathbf{Set}^{\text{op}}}$.

Let I be a ring with $i \in I$ so that $\mathcal{O}$ is an I -indexed category. Then, let $j \in I$, with $j \neq i$, so that there is a *bundle isomorphism*:

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{O}_i \\ \downarrow & \nearrow \sim & \\ \mathcal{O}_j & & \end{array}$$

which induces the following natural transformation

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{\pi_i} & \mathcal{O}_i & & \mathcal{M}_{\mathcal{O}_i} \\ \pi_j \downarrow & \nearrow \sim & \eta \Rightarrow & & \downarrow N \\ \mathcal{O}_j & & & & \mathcal{M}_{\mathcal{O}_j} \end{array}$$

on measurements.

Theorem 1.1. *The map η , which induces a natural transformation on measurments from the bundle isomorphism, exists if and only if the above isomorphism of base spaces are parameterized by the same Heyting algebra $\mathfrak{H}$.*

Proof. Let $\mathfrak{H}$ be a finite truncation of $\mathbb{R}^+$; then, the truncation functor

$$\tau_{\mathbb{I}} : \mathbb{R}^+ \rightarrow \mathbb{I}$$

is a forgetful map, where $\mathbb{I}$ is the disjoint union of intervals $I_0 \sqcup I_1 \sqcup \dots \sqcup I_n$. Then, we can re-write $\mathfrak{H}$ as $\tau_{\mathbb{I}} \circ \mathfrak{H}^+$. Due to the Kochen-Specker theorem, we can always find some $m > n$ so that I_m is outside of $\mathbb{I}$, and therefore the fiber $\pi_i^{-1}(p_m) = \pi_j^{-1}(p_m)$ contains missing sections.³ $\square$

²One might object that since Heyting algebras are, by definition, complete lattices (i.e., they admit all joins and meets), every subalgebra ought to embed into a larger one with suitable extensions. However, this overlooks the semantic distinction between syntactic completion and *global extendability of valuations*. The key point is that while $\mathfrak{H}$ may be complete in the abstract, the measurement scenario imposes additional topological and logical constraints—such as contextual compatibility—that are not captured by lattice-theoretic completeness alone. Specifically, the obstruction we diagnose is not a failure of formal algebraic inclusion but a failure of *coherent gluing*: local sections (contextual valuations) do not always amalgamate into a global section compatible with all contexts. This reflects the topos-theoretic insight that the existence of global elements is a higher-order condition, sensitive to the sheaf (or presheaf) structure of the measurement bundle, not merely to the logical completeness of the underlying algebra.

³A clarification is in order regarding the use of the term “missing sections” in the context of contextuality. We agree that local sections—i.e., consistent valuations over individual measurement contexts—are always definable by construction. However, our usage of the term “missing” refers specifically to the *failure of these local sections to glue into a global one*, as captured by the Kochen–Specker theorem. That is, while each fiber $\pi_i^{-1}(p_i)$ may locally admit a valuation (section), the bundle π over the total configuration space fails to be globally sectionable when measurement contexts are mutually incompatible. The obstruction is not to the existence of local sections per se, but to their *coherence* under gluing. This aligns with standard sheaf-theoretic interpretations in topos quantum theory (cf. [4], [7]), where contextuality is precisely the failure of the global section condition in the spectral presheaf.

One may naturally be left to wonder what this implies realistically for the practicing physicist. In this paper, we explore this issue by considering a very concrete toy model of $(1+1)$ -dimensional topological quantum field theory (*TQFT*) which incorporates contextuality. The reader is recommended to digest the contents of [9, 10] before reading this one; however, this is not a strict pre-requisite. The material here will be as self-contained as possible.

2 Basics of Contextuality

Contextuality can be framed as the existence of variables whose measurements depend upon the outcomes of others; in other words, the existence of non-trivial *marginals*. That is to say:

$$\exists i \exists j \text{ s.t. } \mathcal{M}_{p_i} \wedge \mathcal{M}_{p_j} \neq \mathcal{M}_{p_i \wedge p_j}$$

where $\mathcal{M}_{p_i \wedge p_j}$ is a measurement taken in a *context-free* scenario and $\mathcal{M}_{p_i} \wedge \mathcal{M}_{p_j}$ are two measurements taken in contexts $\mathbb{C}_i$ and $\mathbb{C}_j$. The contexts associated with i and j are called *marginal* if and only if they are non-maximal contexts. The image of a map $\mu : \mathcal{M}_{p_\bullet} \rightarrow \mathbb{R}$ is called the *outcome* of a measurement $\mathcal{M}$ on $p_\bullet$. The outcome of a measurement is a Borel subset of a σ -algebra encoding probabilistic data.

Contextuality was first investigated by Bell and Kochen-Specker in the 1960s [1, 2] to explain the fundamental differences between classical and quantum physics. This led to many seemingly paradoxical conclusions. For instance, Kaufherr and Aharonov [11] concluded that, under classical assumptions, one should in principle not even be able to make a consistent measurement on a particle with one degree of freedom. This has in turn led to dramatic rethinking of quantum theory (*QT*), which has spurred the advent of topos-based formulations of QT.

In particular, the *Bohr topos* emerges as a key construction in topos quantum theory. Inspired by Niels Bohr's doctrine of classical concepts, this approach replaces the traditional Hilbert space formalism with a topos—a category of sheaves or presheaves—that encodes varying classical perspectives (contexts) on a quantum system. Concretely, the Bohr topos is defined as the topos of presheaves over the poset $\mathcal{C}(A)$ of commutative unital C^* -subalgebras of a noncommutative algebra A of observables. Each object in this topos encodes information across all classical contexts simultaneously, while internal logic allows for reasoning in a context-dependent but globally coherent fashion. In this setting, the spectral presheaf replaces the notion of a state space, and contextuality arises from the failure of global sections [7]—there is no consistent global valuation across all contexts.

The two lead of authors of the present paper have tried to reformalize contextuality in the following way: given an interval $\mathbb{I}$ endowed with a gradation (encoded by a sub-object classifier Ω), the map

$$\Omega_{\mathbb{C}_i}(p) : \mathcal{V} \rightarrow \mathbb{I}$$

expresses the measurement outcome. In a context-free setting, we should have

$$\Omega_{\mathbb{C}_i}(p) = \Omega_\bullet(p_i)$$

so that the truth value assigned to p in the context $\mathbb{C}_i$ is equivalent to the truth value assigned to the same measurement on p_i under any other context. This means that in a context-free scenario, index i specifies a unique and manifestly invariant degree of freedom which can be assigned either to a maximal context or to the (body, field, frame of reference...) p itself.

In the language of modal logic, this translates to saying:

$$\forall i \forall p \forall \mathbf{C}_\bullet \quad \exists i \implies \Box \Omega_{\mathbf{C}_\bullet}(p_i)$$

where

$$\Box \Omega_{\mathbf{C}_\bullet}(p_i) = \bigwedge_{0 < j < i} \Diamond p_j$$

is the *joining* of all possible subcontexts. Contextuality violates precisely this condition.

3 A Concrete TQFT that Invokes Contextuality

Let $\mathbf{Cob}_2$ be the oriented *2-dimensional cobordism category*. It provides the natural domain for two-dimensional topological quantum field theories (TQFTs) as described by Atiyah’s axioms [12].

The objects are closed, oriented 1-dimensional manifolds. In dimension one, these are precisely disjoint unions of oriented circles:

$$\Sigma = \bigsqcup_{i=1}^n S^1.$$

where each object Σ is interpreted as a spatial boundary of a physical system at a fixed time slice, and morphisms are oriented 2-dimensional cobordisms $M : \Sigma_1 \rightarrow \Sigma_2$ are compact, oriented surfaces whose boundary decomposes as

$$\partial M = \bar{\Sigma}_1 \sqcup \Sigma_2,$$

where $\bar{\Sigma}_1$ denotes Σ_1 with reversed orientation.

Examples of morphisms:

Cobordism M	Diagrammatic Interpretation	Traditional Physical Meaning
$S^1 \sqcup S^1 \rightarrow S^1$	Pair of pants	Fusion of two particles
$S^1 \rightarrow S^1 \sqcup S^1$	Cobranching	Branching of a single system
$S^1 \rightarrow S^1$	Cylinder	Identity/time evolution
$\emptyset \rightarrow S^1$	Cap	Creation of a state
$S^1 \rightarrow \emptyset$	Cup	Annihilation of a state

These cobordisms represent spacetime evolutions between input and output boundaries. For instance, if we ignore the “traditional” physical interpretations, we can think of a cobranching from one closed submanifold to two others as a bifurcation of a Hilbert space $\mathcal{H}_{S^1_{\text{in}}}$ to a pair $\mathcal{H}_{S^1_{\text{out},(1)}} \otimes \mathcal{H}_{S^1_{\text{out},(2)}}$. If the tensor product fails to be additive, in the sense of projections⁴, then we have a clear case of contextuality. We shall call the first Hilbert space the *necessary Hilbert space* and denote it by $\mathcal{H}_\square$, and the others will be called *possible Hilbert spaces* and shall be denoted by $\mathcal{H}_{\Diamond_\bullet}$ for $\bullet \in \{1, 2\}$.

⁴Here, “additive” refers to the preservation of projection structure across tensor components; e.g., whether $P_{A \otimes B} = P_A \otimes P_B$ holds for compatible projectors.

3.1 Contextuality and Modality for Hilbert Spaces

Let us step back for a moment and recall that we defined measurement scenarios as I -indexed fiber bundles

$$\pi_i : \mathcal{C} \rightarrow \mathcal{O}_i$$

Now, we want to consider them as *cobordisms*. We will think of each $\mathcal{O}_i$ as being the disjoint union of two strata, which correspond to the underlying manifolds of the two possible Hilbert spaces. Then, the cobordism becomes:

$$\Sigma : \mathcal{H}_\square \rightarrow \mathcal{H}_{\diamond,(1)} \sqcup \mathcal{H}_{\diamond,(2)}$$

We will label the sections $\pi_i^{-1}(p_i)$ as $\pi_{i,(1)}^{-1}$ and $\pi_{i,(2)}^{-1}$ depending upon which stratum they lie in. Now, the Kochen-Specker theorem is interpreted as saying:

Theorem 3.1. (*TQFT Kochen-Specker*) *A cobordism is contextual if and only if for some time evolution operator $\mathfrak{U} :: \Sigma$ acting on a necessary Hilbert space $\mathcal{H}_\square$, the resulting scenarios are mutually incompatible.*

Or, equivalently:

Theorem 3.2. *In a contextual TQFT, $\bar{\Sigma}$ is not well-defined.*

Although this appears to be a surprising consequence, it has an intuitive modal explanation: not every possible state can be fibered in the topos of necessary frames; i.e., if we have two differing possible worlds, then at least *one of them* will be missing sections in the necessary world.

3.2 A Minimal Contextual Cobordism Example

Setup: Three Contexts and a Cobranching Cobordism

Let the observables in $\mathcal{C}_\square$ be denoted by $\{A, B\}$, and suppose they satisfy the constraint:

$$AB = -BA.$$

Let each possible context contain one of these observables:

$$\begin{aligned} \mathcal{C}_{\diamond_1} &= \{A\}, \\ \mathcal{C}_{\diamond_2} &= \{B\}. \end{aligned}$$

Suppose further that A and B cannot be simultaneously diagonalized — i.e., there exists no global Hilbert space in which they are both sharp [13] Then no assignment $\phi : A \mapsto a, B \mapsto b$ can be extended from $\mathcal{H}_{\diamond_1}$ and $\mathcal{H}_{\diamond_2}$ back to $\mathcal{H}_\square$ while preserving spectral constraints.

Contextual Failure of Cobordism Reversal

The orientation-reversed cobordism $\bar{\Sigma} : \mathcal{H}_{\diamond_1} \sqcup \mathcal{H}_{\diamond_2} \rightarrow \mathcal{H}_\square$ would represent a recombination of possible states into a necessary one. However, due to the contextual incompatibility between A and B , this reversed process is not definable.

In other words:

- Each context admits a local section (valuation), i.e., $\Box_{\mathcal{C}_{\Diamond_i}}(\phi_i)$ exists,
- But the pullback to $\mathcal{C}_{\Box}$ fails: $\neg\Box_{\mathcal{C}_{\Box}}(\phi)$.

This is a precise modal obstruction. The measurement bundle over Σ cannot be globally sectioned, and thus the monoidal structure breaks:

$$Z(\Sigma) \not\cong Z(\mathcal{H}_{\Diamond_1}) \otimes Z(\mathcal{H}_{\Diamond_2}).$$

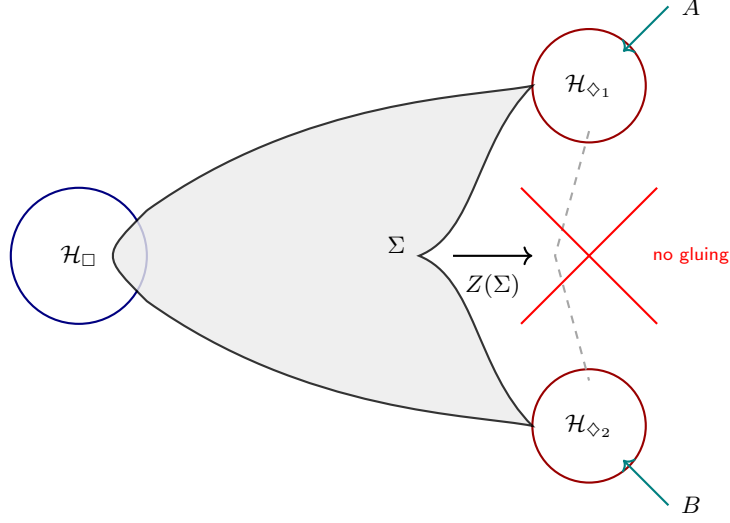


Figure 1: A modal cobordism Σ from the necessary Hilbert space $\mathcal{H}_{\Box}$ to the pair $\mathcal{H}_{\Diamond_1} \sqcup \mathcal{H}_{\Diamond_2}$. Observables A and B obstruct gluing, making the reversed cobordism undefined.

Summary

This simple example illustrates how a cobranching cobordism in $\mathbf{Cob}_2$ can encode a measurement bifurcation whose branches are locally consistent but globally incompatible. Contextuality is diagnosed by the failure of reversal and the non-triviality of the gluing problem. This structure motivates the introduction of **MeasCob**, where such obstructions are treated as native geometric features.

A Future Outlook

In this paper, we merely *diagnosed* and explained the “problem” of contextuality. It is an open question whether we can actually treat the problem. For future works, we propose five research directions:

- **Extension to higher-dimensional TQFTs:** Can similar contextual obstructions be characterized in $\mathbf{Cob}_n$ for $n > 2$?

- **Cohomological formulations:** Is there a cohomological class (e.g., via Čech cohomology or sheaf cohomology) that captures contextual obstructions?
- **Functorial quantization:** Can we define a functor $Z : \text{MeasCob} \rightarrow \text{Vect}_{\mathbb{C}}$ that systematically accounts for contextual failures?
- **Computational models:** Can these categorical structures be used to simulate or classify contextual phenomena in quantum computation?
- **Ultranaughtical Observables:** Can we diagnose specific discrepancies between measurement contexts using ultranaught scenarios?

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C A.I. Disclaimer

Artificial intelligence was used for the following purpose(s): generation of tables; correction of typos.

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